MVT $\frac{f(b)-f(a)}{b-a}$ $\frac{x_1-x_0}{b-a}$ $f(b)-f(a)$ $x_n - x_{n-1}$

university of groningen

Exam Computer Assisted Problem Solving (CAPS)

June 27th 2019 9.00-12.00

This exam is 'closed book'. It is NOT allowed to use a book or lecture notes. Only the use of a simple pocket calculator is allowed. Programmable calculators are not allowed, nor the use of electronic devices (tablet, laptop, etc.).

Always give a clear explanation of your answer. An answer without any computation will not be rewarded, so also copy the computations from your scratch paper.

Write your name and student number on each page!

Free points: $\boxed{10}$

1. The equation $e^{-x} = 5x - 10$ has a solution near $x = 2.0$.

   (a) $\boxed{7}$ (1) Compute one iteration with the Secant method, starting with $x_0 = 2.0$, $x_1 = 2.1$.
   (2) Determine the most accurate ('the best') error estimate for x_2.
   (3) What can you say regarding the number of iterations required for an accuracy of 1.0E-9 ? -

   (b) $\boxed{4}$ Determine a $x_{n+1} = g(x_n)$ method with optimal linear convergence factor for this problem, by introduction and optimisation of a parameter α.

   (c) $\boxed{7}$ When $x_{n+1} = \frac{1}{5}e^{-x_n} + 2$ is used, again with $x_0 = 2$, the first iterations are given by

x_0	x_1	x_2	x_3	x_4
2.00000000	2.02706706	2.02634426	2.02636331	2.02636280

   (1) How fast does this method converge? Give order and factor of the error reduction.
   (2) Determine an error estimate for x_4.
   (3) Calculate an improved solution by means of Steffensen extrapolation.

   (d) $\boxed{8}$ Give a complete program (in pseudo code or Matlab-like language) that solves the problem with an accuracy of tol=1E-6, using the Newton method. Use an appropriate stopping criterion. ±

   check

2. The value of the n-th order Bessel function $J_n(x)$ at $x = 1$ is given by
$$J_n(1) = \frac{1}{\pi}\int_0^{\pi} \cos(n\,u - \sin(u))\,du.$$

   (a) $\boxed{3}$ Suppose $J_0(1)$ is computed on a grid with only one segment $[0, \pi]$. Does 1st-order interpolation, as part of the Trapezoidal method, yield a better accuracy than the constant interpolation of the Midpoint method, in this case? Explain why. -

   (b) $\boxed{8}$ (1) Use the Trapezoidal method on a grid with two segments to approximate $J_0(1)$.
   (2) Give an error estimate for the result at (1) using the global error theorem.
   Hint: you may directly use that on $[0, \pi]$ the function $f''(x)$ has extreme values at $x=0$, $x=\pi/2$ and $x=\pi$, without considering $f'''(x)$.

   (c) $\boxed{7}$ When n is not an integer, e.g. $n = 1/2$, manual computation becomes difficult. With the Trapezoidal method on finer grids the following results are obtained for $J_{1/2}(1)$

   $I(n)$ is the approximation of the integral on a grid with n sub-intervals.

n	$I(n)$
16	0.85362844
32	0.85478163
64	0.85506942
128	0.85514134

   (1) Compute the q-factor. What can you conclude?
   (2) How many segments are required (use powers of 2) for an accuracy of 1.0E-8?
   (3) Compute an improved solution for $I(128)$ through T_2 extrapolation. Does it make sense to further extrapolate into $T_3(128)$? Explain. -

P.T.O.

$\alpha=2$

(c) $\boxed{8}$ Give a complete program (in pseudo code or Matlab-like language) that solves the problem with accuracy tol=1E-6, using the Midpoint method (no extrapolation). Use an appropriate error estimate for the stopping criterion. +−

3. Consider on [0; 5] the o.d.e. $y'(x) = \frac{1}{10}y^2(x) + \alpha$, with boundary condition $y(0) = 2$.

(a) $\boxed{9}$ Take $\alpha = -\frac{2}{10}$. Use a grid with Δx=0.5.

(1) Use explicit Euler to compute $y(x)$ at x=1 (these are 2 steps).

(2) Use Heun's method (RK2) to compute $y(x)$ at x=0.5.

(3) Use the Crank Nicolson (i.e. Trapezoidal) method to compute $y(x)$ at x=0.5. −

$\alpha=2$

(b) $\boxed{9}$ For $\alpha = -\frac{1}{10}$, Heun's method is used on a number of grids (N=10, 20, 40 segments). The table below shows solutions at a selection of x locations.

x_n	$N = 10$	$N = 20$	$N = 40$
2	0.39312040	0.36681244	0.35888496
3	-0.58166323	-0.61912718	-0.62994516
4	-1.44495774	-1.49233433	-1.50488101
5	-2.09054035	-2.14428082	-2.15717831

(1) Is there a stability limit visible? Explain. − $|1+ah| \leq 1$ −

(2) Compute the q-ratio for x=4. What can you conclude?

(3) Give error estimates ϵ for the solutions at $x = 3$ and $x = 4$ on the finest grid. Which value is the larger one? Explain how come.

(4) Compute an improvement for the solution at $x = 4$ by means of extrapolation.

(5) Theoretically, the extrapolation in (4) has 4th order accuracy, like RK4. Give both an advantage and a disadvantage of RK2 extrapolation compared to RK4. −

(c) $\boxed{8}$ Give a complete program (in pseudo code or Matlab-like language) that solves the problem with accuracy tol=1E-6, using the Heun method (without extrapolation). Use an appropriate error estimate for the stopping criterion. +−

$\alpha=2$

4. Consider the diff. eqn. $y''(x) + \alpha\, y'(x) + 10\, y(x) = 2\cos(\pi\, x)$, with boundary conditions $y(0) = 1$ and $y(2) = 0$.

(a) $\boxed{4}$ Take $\alpha = 0$, such that $y'(x)$ is (temporarily) out of the diff. eqn. Give the matrix and rhs-vector when the problem is solved on a grid with $N = 4$ segments by means of the matrix method, using the [1 -2 1]-formula for $y''(x)$.

(b) $\boxed{2}$ Which modification do you have to make to the system when the boundary condition at $x = 0$ is changed into $y'(0) = 2$?

(c) $\boxed{6}$ The boundary condition at $x = 0$ is switched back to $y(0) = 1$. For $\alpha \neq 0$, the term $y'(x)$ can be approximated by means of $y'(x_i) = \dfrac{y_i - y_{i-1}}{\Delta x}$.

(1) Which modification do you have to make to the system when $y'(x)$ is treated in this way? Give the matrix and rhs-vector when $N = 4$, $\alpha = 2$.

(2) Describe the influence of the boundary value $y(0) = 1$ on the solution at $x = 1$, for the case $N = 4$, $\alpha = 2$.

(3) On a grid with $N = 100$, will the solution for $\alpha = 2$ be more accurate than for $\alpha = 0$? Explain.

Total: $\boxed{100}$